# The Oscillator as a Dynamical System

Miller Puckette
Department of Music, University of California, San Diego, USA
msp [at] ucsd.edu
https://msp.ucsd.edu

Proceedings of Korean Electro-Acoustic Music Society's 2017 Annual Conference (KEAMSAC2017)
Gwangju, Korea, 26-28 October 2017

The oscillator is the most generally useful and fundamental tool used in electronic music. It is an example of a dynamical system in which the overall state of a system, often described as a small number of state variables, evolves in time in a way that can make interesting musical sounds. This talk examines some classical and novel ways in which a dynamical system can give rise to novel acoustical behaviors that can be used musically. Examples include strategies for coupling two or more oscillators together, and billiard-ball trajectories on non-rectangular tables.

Solid-state and vacuum-tube audio circuits may be considered as having a state that changes in time. The state consists of the time-varying values of the charges of all the individual capacitors (and inductors, if any). This description is a slight idealization since other circuit elements also have state but at audio rates they usually may be idealized as stateless, leaving just a relatively small number of time-varying parameters. It is often possible to describe such a circuit as an explicit system of differential equations:

$$x'_1 = f_1(x_1, \ldots, x_N, t)$$
$$\cdots$$
$$x'_N = f_N(x_1, \ldots, x_N, t)$$

Here, $x_1, \ldots, x_N$ are the charges of the $N$ capacitors, and the functions $f_1, \ldots, f_N$ describe how each them change in time, depending on their current values and also possibly depending on the current time $t$. For example, a passive low-pass filter realized with a resistor and a capacitor (an RC circuit) can be described this way:

$$x'(t) = a \cdot (u(t) - x(t))$$

where the single state variable $x$ is the output of the filter, $u(t)$ is the filter's input, and $a$ is a parameter that determines the roll-off frequency of the filter.

There is a well-known approach to analyzing the behavior of this particular example using linear systems theory, and it is not my intention to claim that the dynamic-systems view of this particular system gives any new insights about it, but instead, this well-known example is a convenient one for illustrating the dynamical systems point of view. Seen this way, the *state space* of the system is a line in which the state variable $x(t)$ travels in time. The velocity of travel is described as a vector field. At each point in the state space the velocity is given by the vector with components $f_1$, ..., $f_n$. This is shown in Figure 1.

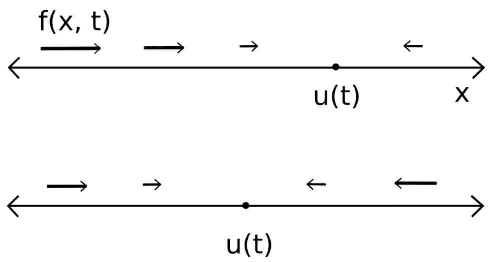


**Figure 1.** Low-pass filter considered as a dynamical system. The input, u(t), varies with time, and as a result the vector field that defines the dynamical system does also. The vector field also depends on the roll-off frequency of the filter.

The vector field defines a *flow* through the state space, and one imagines a particle floating in the state space and describing a *path* through the state space with the passage of time. Although this formulation applies to systems that evolve continuously in time, it can be used in discrete-time digital systems by approximation. This is one approach, for example, that has been taken to model analog circuits such as the Moog ladder filter (Huovilainen 2004). Two recent PhD graduates of UCSD have explored ways of using continuous-time models to give rise to design new digital musical instruments (Allen 2014 ; Medine 2016).

The functions defining the vector field (flow) may depend on time or not. In the above example the flow does depend on time, and this dependence is determined in real time as a result of the system's input.

## Oscillator as two-dimensional flow

As another preliminary example, a nonlinear harmonic oscillator may be realized as a two-variable dynamical system whose state evolves in time according to these equations:

$$x'_1 = -k x_2 + (1 - x_1^2 - x_2^2) x_1$$

$$x'_2 = k x_1 + (1 - x_1^2 - x_2^2) x_2$$

The associated vector field is shown in Figure 2.

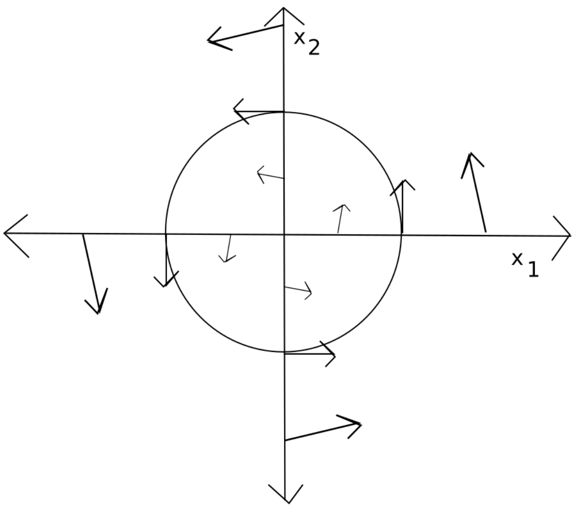


**Figure 2.** A sinusoidal oscillator realized as a two-dimensional dynamical system. The circular path is the limit cycle of the oscillator. Points that do not lie on the circle (except the origin) spiral inward or outward to the limit cycle.

This type of oscillator may be forced by one or two other time-varying functions by adding them to the expressions above, and this can give rise to some interesting non-linear behaviors. As a self-contained system, though, it has a limited range of potential behaviors. In general, dynamical systems whose phase space is a line or plane cannot exhibit the chaotic behaviors for which they are usually studied and used.

The best-known example of a chaotically behaving dynamical system is the *Lorenz attractor* whose phase space is three dimensional. However, as we will see in the next section, non-periodic, apparently chaotic behavior can also be realized in two-dimensional phase spaces; but for this to be possible their topology must be richer than that of a plane or sphere.

## Coupled oscillator pair

To properly enjoy the following treatment the reader is invited to listen to a piece by Pauline Oliveros, *Bye Bye Butterfly* (1965), realized in the San Francisco Tape Music Center using equipment designed by Donald Buchla. The synthetic sounds are made using an "oscillator" (as Oliveros described it) but to the ear it is more than a simple oscillator and is probably an early version of Buchla's later dual oscillator module in which one oscillator is synchronized by a second one. The circuit's behavior is quite complex and hard to describe completely, but a simple version can be constructed abstractly as a dynamical system with a two-dimensional state space which is topologically a torus.

To synchronize one oscillator (call it A) to another one (B), we assign each of them its own frequency, but whenever B reaches a particular phase in its cycle (which we can label as "phase zero"), we reset the phase of A. In the simplest case, called *hard synchronization* (or "hard sync"), the phase of A is always set exactly the same (also to zero as we will consider it) each time B cycles, so that the result has the periodicity of B (although the timbre depends strongly on the frequencies of both A and B).

In the more complicated and interesting case called *soft synchronization*, the phase of A is made to move quickly toward zero when B's phase passes through zero, but not so quickly as to necessarily reach it. The phase of A may be chosen always to move forward, always backward, or in a direction that depends on the phase itself; for instance, forward if more than halfway to the next zero phase or backward otherwise.

The two-oscillator system can be set up as a dynamical system with only two state variables, equal to the phases of the two oscillators. If the two oscillators are each running freely (not coupled or synced together) the vector field is uniform (constant) over the entire phase space, as shown in Figure 3.

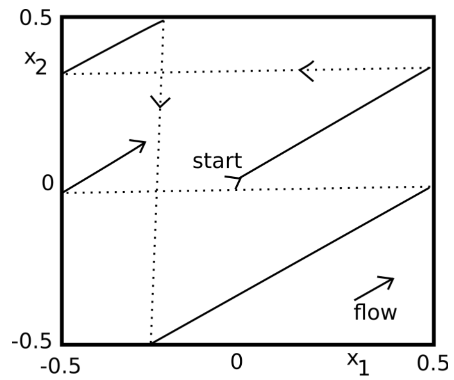

**Figure 3.** Phase space representation of a pair of uncoupled oscillators. Each phase ranges from -0.5 to 0.5. The vector field (the "flow" is uniform. Its two components are the frequencies of the oscillators.

The state of the pair of oscillators propagates along a straight path that wraps around each time it reaches an edge of the square. (More precisely, the two sides of the phase space should be considered as the same points, and the top should be considered as the same as the bottom. Seen this way, the phase space is a torus and the path is continuous since the dotted segments each extend from a point to itself.)

We can now introduce interactions between the two oscillators by making the flow non-uniform, so that the magnitude or direction of the flow differs from point to point. For example, to soft-synchronize the oscillator A (whose phase is $x_1$ ) to B, we can change the frequency of A to aim it toward phase zero in a region where the phase of B is near zero. Outsize this region the flow remains uniform as before. This is shown in Figure 4.

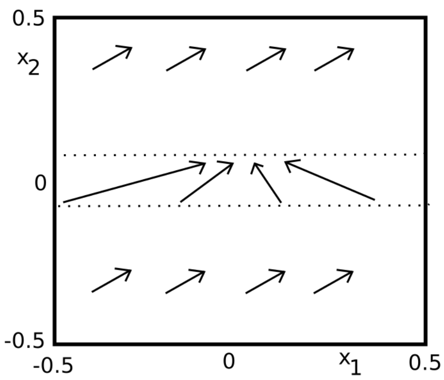


**Figure 4.** A Soft-synchronized oscillator pair as a dynamical system whose phase space is a torus. Inside the region between the two dotted lines the phase of the first oscillator is pushed toward zero.

This system can make a range of interesting and musically useful, non-periodic sounds. The sounds vary according to: the frequencies of the two oscillators, the size of the region in which the second oscillator affects the first, the strength of the correction of the phase, and when and whether to push the phase of the first oscillator forward or backward toward zero. Also, it is straightforward to generalize the system further. For example, the two oscillators can each be soft-synchronized to the other, and/or there may be three or more oscillators in the system.

## Wormholes in flat space

Another possible approach was developed in an earlier paper of mine (Puckette 2015), Here I'll describe it in a simpler and more useful form than I was able to at the time. We start again with a two-dimensional phase space configured as a torus, but instead of making the flow non-uniform within it, we introduce one or more wormholes in the space. These are areas that are not used but, instead, directly jumped over whenever the path reaches them. One of the simplest possibilities is to specify a rectangular region in the phase space, arranging the phase so that the rectangle is centered as shown in Figure 5. When the path (which is assumed to be traveling in the northeast direction) arrives at an edge of the rectangle, it jumps discontinuously to the opposite point and then continues as before. Conceptually this need not be considered a discontinuity because we can suppose that the space itself is connected together in this way.

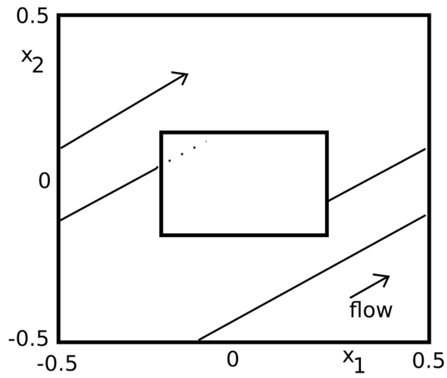


Figure 5. A phase space with a rectangular wormhole in the shape of a rectangle. The vector field is uniform, but whenever the state would enter the wormhole, instead it jumps to the diametrically opposite point.

As with the previous system, this one can readily be generalized to more dimensions. This is the sound source for a collaborative piece I perform with composer Kerry Hagan titled *Hack Lumps*, using three dimensions and six numerical controls to set each of the three frequencies and the three dimensions of the wormhole (which is then in the shape of a rectangular box).. If any dimension of the box is zero the three oscillators sound independently, and their interconnectedness increases (and the stability of the sound typically decreases) as the size of the box is increased.

## Triangular pool table

Taking the pair of oscillators as a point of departure again, we proceed in a different direction by analogy with the trajectory of a billiards (pool) ball on a table in the shape of an isosceles right (45-45-90-degree) triangle. Assuming perfect reflections and no friction, the ball takes a path that is the same as that of the two independent oscillators, as can be seen by dividing the square representing the phase space into eight octants as shown in Figure 6. The solid path shows a reflecting path within one of the eight triangles, corresponding to a straight-line path through a phase space eight times as large. Depending on which of the eight triangles a point is in, we can deduce not only where it is within the triangle, but also in which of eight possible directions it is traveling (assuming the 8 reflected directions are distinct.)

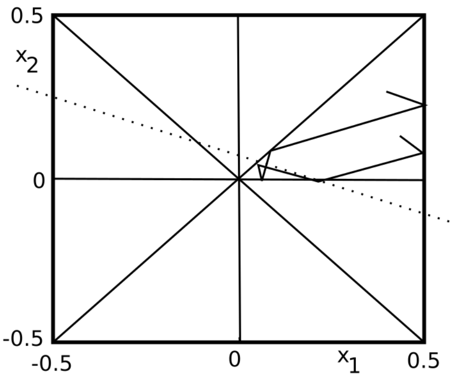


**Figure 6.** A Soft-synchronized oscillator pair as a dynamical system whose phase space is a torus. The dotted line shows the path through phase space and the solid line shows the corresponding path through one of the triangles.

If, however, the triangle has any other shape (except equilateral, in which case a similar reflection argument holds), the system becomes much harder to analyze and start to exhibit chaotic behavior. As in the case of the coupled oscillators, the chaotic behavior can be made to closely resemble the uncoupled case or diverge wildly from it at will.

The phase space here is three-dimensional: to understand where we are in the system we need to know the position (two independent coordinates) and the velocity vector (one coordinate because the speed remains constant and only the direction varies). Points on the edge of the triangle (with a given direction) are identified with the same point and the reflected direction.

## Realizing the sound

In general, an oscillator's phase is not heard directly as sound, but instead determines the location in a waveform. In cases where the phase space is higher-dimensional, we can use any function of the phase space (although there will in general be smoothness conditions to avoid foldover if the output is a sampled audio signal.) In *Hack Lumps* (using the wormhole technique), we chose to output three signals, each an even function of one of the three phases, equal to zero over the entire wormhole region (the function is therefore changed as a function of the size of the wormhole).

In all the cases shown, a powerful simplification is available that relieves us of the necessity to numerically solve the differential equations, since all the paths are straight lines, traveled at uniform velocities, except at the points where they intersect a boundary (which itself is a straight line or a plane in our examples.) We need only make one computation each time a boundary is reached, to find the new velocity and the time at which the next boundary is to be hit, and then interpolate linearly between the breakpoints obtained this way. In some cases (particularly the triangular pool table) more than one breakpoint may be hit between two consecutive audio samples.

**Acknowledgments.** *In memoriam*, Donald Buchla and Pauline Oliveros.

## References

Allen, A.S. (2014). "Ruratae: a physics-based audio engine," Ph.D. dissertation, University of California, San Diego.

Huovilainen, A. (2004). "Non-linear digital implementation of the Moog ladder filter," in Proceedings of the International Conference on Digital Audio Effects. (DAFx-04), pp. 61–64.

Medine, D. (2016). "Dynamical systems for audio synthesis: Embracing nonlinearities and delay-free loops," Applied Sciences, vol. 6, no. 5, article number 134.

Puckette, M. (2915) "The Sampling Theorem and its discontents." ICMC Keynote talk, reprinted in Array Magazine, 2017.